

Weakly decomposable regularization penalties and structured sparsity

Weakly decomposable regularization penalties

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Abstract It has been shown in literature that the Lasso estimator, or ℓ_1 -penalized least squares estimator, enjoys good oracle properties. This paper examines which special properties of the ℓ_1 -penalty allow for sharp oracle results, and then extends the situation to general norm-based penalties that satisfy a weak decomposability condition.

Key words: Lasso, sharp oracle inequality, weakly decomposable norm, sparsity, regularization

1 Introduction

The Lasso (Tibshirani [1996]) has become extremely popular in the last several years. It is a computationally tractable method for high-dimensional models, with good theoretical properties. Several types of modifications of the Lasso have been introduced and studied, such as the fused Lasso (Tibshirani et al. [2005]) and the smoothed Lasso (Hebiri and van de Geer [2011]). In this paper, we are primarily interested in extensions of the ℓ_1 -penalty to general structured sparsity penalties such as the group Lasso introduced by Yuan and Lin [2006]) and further structured versions given by Zhao et al. [2009], Jacob et al. [2009], Jenatton et al. [2011] and Micchelli et al. [2010]. We will provide sharp versions of the oracle inequalities given in Bach [2010] and extend the sharp oracle results for the Lasso and nuclear norm penalization as given in Koltchinskii et al. [2011] and Koltchinskii [2011] to general structured sparsity penalties, where we in addition prove inequalities for the estimation error.

Consider the linear model

$$Y = X\beta^0 + \epsilon,$$

where Y is an n -vector of observations, X is a given $n \times p$ matrix, ϵ is an n -vector of errors and β^0 is a p -vector of unknown coefficients. The Lasso estimator is

$$\hat{\beta} := \arg \min_{\beta \in \mathbb{R}^p} \left\{ \|Y - X\beta\|_n^2 + 2\lambda \|\beta\|_1 \right\}.$$

Here, $\|\beta\|_1 := \sum_{j=1}^p |\beta_j|$ denotes the ℓ_1 -norm of the vector β and for a vector $v \in \mathbb{R}^n$ we let $\|v\|_n$ be the normalized Euclidean norm $\|v\|_n := \sqrt{v^T v / n}$. Finally $\lambda > 0$ is a tuning parameter. The ℓ_1 -penalty is a variable selection or (soft-)thresholding type penalty: the larger λ , the more coefficients $\hat{\beta}_j$ will be set to zero.

In this paper, we first briefly review a sharp oracle result of Koltchinskii et al. [2011] for the Lasso estimator. We then extend the sharp oracle result to other norm-penalties, satisfying a weak decomposability condition as given in Section 4.

The paper is organized as follows. We first introduce the concept “effective sparsity” in Section 2. Effective sparsity plays a crucial role in all our results. As a benchmark, we then restate in Section 3 an oracle inequality from

Koltchinskii et al. [2011] for the Lasso. Theorem 4.1 in Section 4 contains the main result. It extends the ℓ_1 -norm penalty to general weakly decomposable norm-penalties. Some examples are given in Section 5. In Section 6 we consider comparison of the effective sparsity based on the ℓ_1 -norm to the effective sparsity based on a different norm. A brief discussion of the results and further research is given in Section 7. Finally, Section 8 contains the proofs.

2 The ℓ_1 -eigenvalue and effective sparsity for the ℓ_1 -norm

To state an oracle result, we need to define the ℓ_1 -eigenvalue $\delta(L, S)$, where $L > 0$ is a constant and $S \subset \{1, \dots, p\}$ is an index set. We use the notation

$$\beta_{j,S} := \beta_j \mathbf{1}\{j \in S\}, \quad j = 1, \dots, p.$$

Thus β_S is a p -vector with zero entries at the indexes $j \notin S$. We will sometimes identify β_S with the vector $\{\beta_j\}_{j \in S} \in \mathbb{R}^{|S|}$.

Definition 2.1 For constant $L > 0$ and an index set S , the ℓ_1 -eigenvalue is

$$\delta(L, S) := \min \left\{ \|X\beta_S - X\beta_{S^c}\|_n : \|\beta_S\|_1 = 1, \|\beta_{S^c}\|_1 \leq L \right\}.$$

The compatibility constant is

$$\phi^2(L, S) := |S| \delta^2(L, S).$$

The geometric interpretation of the ℓ_1 -eigenvalue, as given in van de Geer and Lederer [2012], is as follows. Let $X_j \in \mathbb{R}^n$ denote the j -th column of X ($j = 1, \dots, p$). The set $\{X\beta_S : \|\beta_S\|_1 = 1\}$ is the convex hull of the vectors $\{\pm X_j\}_{j \in S}$ in \mathbb{R}^n . Likewise, the set $\{X\beta_{S^c} : \|\beta_{S^c}\|_1 \leq L\}$ is the convex hull including interior of the vectors $\{\pm L X_j\}_{j \in S^c}$. Thus, the ℓ_1 -eigenvalue $\delta(L, S)$ is the distance between these two sets. We note that:

- if L is large the ℓ_1 -eigenvalue will be small,
- it will also be small if the vectors in S exhibit strong correlation with those in S^c ,
- when the vectors in $\{X_j\}_{j \in S}$ are linearly dependent, it holds that

$$\{X\beta_S : \|\beta_S\|_1 = 1\} = \{X\beta_S : \|\beta_S\|_1 \leq 1\},$$

and hence then $\delta(L, S) = 0$.

The compatibility constant was introduced in van de Geer [2007]. Its name comes from the idea that when $\phi(L, S)$ is large the normalized Euclidean norm $\|\cdot\|_n$ and the ℓ_1 -norm $\|\cdot\|_1$ are in sense compatible. The difference between the compatibility constant and the squared ℓ_1 -eigenvalue lies only in the normalization by the size $|S|$ of the set S . This normalization is inspired by the orthogonal case, which we detail in the following example.

Example 2.1 Suppose that the columns of X are all orthogonal: $X_j^T X_k = 0$ for all $j \neq k$. Assume moreover the normalization $\|X_j\|_n = 1$ for all j . Then clearly,

$$\|X\beta_S - X\beta_{S^c}\|_n = \|\beta_S - \beta_{S^c}\|_2,$$

where $\|\beta\|_2 := \sqrt{\sum_{j=1}^p \beta_j^2}$ is the ℓ_2 -norm of the vector β . But

$$\|\beta_S - \beta_{S^c}\|_2^2 = \|\beta_S\|_2^2 + \|\beta_{S^c}\|_2^2 \geq \|\beta_S\|_2^2 \geq \|\beta_S\|_1^2/|S|,$$

and in fact

$$\min_{\|\beta_{S^c}\|_1 \leq L, \|\beta_S\|_1=1} \|\beta_S - \beta_{S^c}\|_2^2 = \min_{\|\beta_S\|_1=1} \|\beta_S\|_2^2 = 1/|S|.$$

It follows that $\delta^2(L, S) = 1/|S|$ and $\phi^2(L, S) = 1$.

A vector β is called sparse if it has only few non-zero coefficients. That is, the cardinality $|S_\beta|$ of the set $S_\beta := \{j : \beta_j \neq 0\}$ is small. We call $|S_\beta|$ the sparsity-index of β . More generally, we call $|S|$ the sparsity index of the set S . The *effective* sparsity, as defined in van de Geer and Müller [2012], takes into account the correlation structure in the design matrix X .

Definition 2.2 For a set S and constant $L > 0$, the *effective sparsity* $\Gamma^2(L, S)$ is the inverse of the squared ℓ_1 -eigenvalue, that is

$$\Gamma^2(L, S) = \frac{1}{\delta^2(L, S)}.$$

In other words, for orthogonal design the effective sparsity of a set S is its cardinality, and in general, it is the inverse of the squared distance between the convex hull $\{X\beta_S : \|\beta_S\|_1 = 1\}$ and the convex set $\{X\beta_{S^c} : \|\beta_{S^c}\|_1 \leq L\}$.

Finally, we give a small numerical example from van de Geer and Müller [2012].

Example 2.2 As a simple numerical example, let us suppose $n = 2$, $p = 3$, $S = \{3\}$, and

$$X = \sqrt{n} \begin{pmatrix} 5/13 & 0 & 1 \\ 12/13 & 1 & 0 \end{pmatrix}.$$

Since the sparsity index is $|S| = 1$, the ℓ_1 -eigenvalue $\delta(L, S)$ is equal to the square root $\phi(L, S)$ of the compatibility constant, and equal to the distance of X_1 to line that connects LX_1 and $-LX_2$, that is

$$\delta(L, S) = \max\{(5 - L)/\sqrt{26}, 0\}.$$

Hence, for example for $L = 3$ the effective sparsity is $\Gamma^2(3, S) = 13/2$.

Alternatively, when

$$X = \sqrt{n} \begin{pmatrix} 12/13 & 0 & 1 \\ 5/13 & 1 & 0 \end{pmatrix},$$

then for example $\delta(3, S) = 0$ and hence $\Gamma^2(3, S) = \infty$. This is due to the sharper angle between X_1 and X_3 .

3 An oracle inequality for the ℓ_1 -norm

For a vector $w \in \mathbb{R}^p$, we let $\|w\|_\infty := \max_{1 \leq j \leq p} |w_j|$ be the uniform norm. The following theorem is a slight extension of Koltchinskii et al. [2011] (we use the effective sparsity instead of restricted eigenvalues). The sparsity oracle inequality in this theorem is a simple consequence of the following properties of the ℓ_1 -norm:

- Dual norm equality: $\sup\{|w^T \beta| : \|\beta\|_1 \leq 1\} = \|w\|_\infty, \forall w$,
- Triangle inequality: $\|\beta + \tilde{\beta}\|_1 \leq \|\beta\|_1 + \|\tilde{\beta}\|_1, \forall \beta, \tilde{\beta}$,
- Decomposability: $\|\beta\|_1 = \|\beta_S\|_1 + \|\beta_{S^c}\|_1, \forall \beta, S$.

Note that the triangle inequality implies convexity: $\|\alpha\beta + (1-\alpha)\tilde{\beta}\|_1 \leq \alpha\|\beta\|_1 + (1-\alpha)\|\tilde{\beta}\|_1, \forall \beta, \tilde{\beta}$ and all $0 \leq \alpha \leq 1$. Convexity of the penalty is crucial for deriving oracle inequalities that are sharp. Lemma 8.1 gives the details.

Recall the notation

$$S_\beta := \{j : \beta_j \neq 0\}, \beta \in \mathbb{R}^p.$$

Theorem 3.1 (Koltchinskii et al. [2011]) *Let for $S \subset \{1, \dots, p\}$*

$$\lambda^S := \|(\epsilon^T X)_S\|_\infty / n, \lambda^{S^c} := \|(\epsilon^T X)_{S^c}\|_\infty / n.$$

Define for $\lambda > \lambda^{S^c}$

$$L_S := \frac{\lambda + \lambda^S}{\lambda - \lambda^{S^c}}.$$

Then

$$\|X(\hat{\beta} - \beta^0)\|_n^2 \leq \min_{\beta \in \mathbb{R}^p, S=S_\beta, \lambda > \lambda^{S^c}} \left\{ \|X(\beta - \beta^0)\|_n^2 + (\lambda + \lambda^S)^2 \Gamma^2(L_S, S) \right\}.$$

Thus, the Lasso trades off an approximation error $\|X(\beta - \beta^0)\|_n^2$ with an estimation error $(\lambda + \lambda^S)^2 \Gamma^2(L_S, S)$. The above oracle inequality is called sharp because the constant in front of the approximation error $\|X(\beta - \beta^0)\|_n^2$ is one. Apart from Koltchinskii et al. [2011] and Koltchinskii [2011], results in literature are mostly non-sharp versions, with a constant larger than one in front of the approximation error, see e.g. Bühlmann and van de Geer [2011]. It is interesting to note that convexity of the penalty plays a crucial role, e.g., with the ℓ_0 -penalty one cannot arrive at sharp oracle results. Observe that we do not present a bound for the ℓ_1 -error in Theorem 3.1. We will show how such a bound can be included in the results in Theorem 4.1.

Remark 3.1 *It is as yet not clear to what extent ℓ_1 -eigenvalue conditions are necessary for oracle behavior of the prediction error $\|X(\hat{\beta} - \beta^0)\|_n^2$ of the Lasso estimator. For example, if the design matrix X has repeated columns (or columns that are proportional) in the set S , then the ℓ_1 -eigenvalue will be zero. A reparametrization argument shows however that the Lasso estimator behaves as if repeated columns are treated as one.*

4 A sharp oracle inequality for general weakly decomposable penalties

Let Ω be some norm on \mathbb{R}^p , and let $\hat{\beta}$ be the norm-penalized estimator

$$\hat{\beta} := \hat{\beta}_\Omega := \arg \min_{\beta \in \mathbb{R}^p} \left\{ \|Y - X\beta\|_n^2 + 2\lambda\Omega(\beta) \right\}.$$

We will derive an oracle inequality for $\hat{\beta}$ for weakly decomposable norms Ω , a notion introduced in Definition 4.1.

Recall that the ℓ_1 -norm is decomposable: $\|\beta\|_1 = \|\beta_S\|_1 + \|\beta_{S^c}\|_1$ for all vectors β and any set S . The triangle inequality of course holds for any norm Ω and so does the dual norm equality with the uniform norm replaced by the dual norm

$$\Omega_*(w) := \sup_{\Omega(\beta) \leq 1} |w^T \beta|.$$

We stress that the triangle inequality and dual norm equality fail to hold if we replace the norm by powers of that norm. For example, the triangle inequality does not hold for $\|\cdot\|_2^2$, which will mean the ridge regression penalty does not fall within our framework. Returning to a general norm Ω , it is not necessarily decomposable. Decomposability is however very useful for the derivation of oracle inequalities, an observation which was discussed previously by van de Geer [2001], van de Geer [2010] (where the property is called separability) and Negahban et al. [2012]. Note that powers of norms can be decomposable, for example $\|\beta\|_2^2 = \|\beta_S\|_2^2 + \|\beta_{S^c}\|_2^2$. However, the required triangle inequality does not hold for $\|\cdot\|_2^2$.

We will show now that decomposability is not a necessary condition for oracle results. This was also realized by Bach [2010], although there the situation is restricted to structured sparsity norms defined by sub-modular functions. We consider general norms Ω , which are perhaps not decomposable, but only *weakly decomposable* for certain index sets S , which means that the norm $\Omega(\beta)$ of an arbitrary vector β is always superior to the sum of norms of β_S and β_{S^c} .

Definition 4.1 *Fix some set S . We say that the norm Ω is weakly decomposable if there exists a norm Ω_{S^c} on $\mathbb{R}^{p-|S|}$ such that for all $\beta \in \mathbb{R}^p$,*

$$\Omega(\beta) \geq \Omega(\beta_S) + \Omega^{S^c}(\beta_{S^c}).$$

Definition 4.2 *We say that S is an allowed set (for Ω) if Ω is weakly decomposable for S .*

The best choice for Ω^{S^c} is to take $\Omega^{S^c}(\beta_{S^c})$ as large as possible (see also Section 7). We identify β_{S^c} with the $(p - |S|)$ -vector $\{\beta_j\}_{j \in S^c}$ and consider Ω^{S^c} as norm on $\mathbb{R}^{p-|S|}$ instead of \mathbb{R}^p . There may be no “natural” extension to a norm on \mathbb{R}^p (see Section 5.3 for an illustration), and an extension is also not needed.

Observe that any norm is trivially (weakly) decomposable for the empty set and for the complete set $\{1, \dots, p\}$ containing the indices of the all variables.

Some examples, where we in particular discuss nontrivial choices of S , will be given in Section 5.

We also extend the definition of ℓ_1 -eigenvalues and effective sparsity to general weakly decomposable norms.

Definition 4.3 *Suppose S is an allowed set. Let $L > 0$ be some constant. The Ω -eigenvalue (for S) is*

$$\delta_\Omega(L, S) := \min \left\{ \|X\beta_S - X\beta_{S^c}\|_n : \Omega(\beta_S) = 1, \Omega^{S^c}(\beta_{S^c}) \leq L \right\}.$$

The Ω -effective sparsity is

$$\Gamma_\Omega^2(L, S) := \frac{1}{\delta_\Omega^2(L, S)}.$$

The Ω -eigenvalue $\delta_\Omega(L, S)$ depends on the choice of the norm Ω^{S^c} , but we do not express this in our notation. It has a similar geometric interpretation as the ℓ_1 -eigenvalue: $\delta_\Omega(L, S)$ is the distance between the sets $\{X\beta_S : \Omega(\beta_S) = 1\}$ and $\{X\beta_{S^c} : \Omega^{S^c}(\beta_{S^c}) \leq L\}$. The shape of these sets depends heavily on the norms Ω and Ω^{S^c} .

We will use the effective sparsity to bound the norm of β_S in terms of $\|X\beta\|_n$, as detailed in the following lemma. Here we use the “cone condition” for Ω .

Definition 4.4 *Let $L > 0$ be some constant, S some allowed set and $\beta \in \mathbb{R}^p$ some vector. We say that β satisfies the (L, S) -cone condition for Ω if $\Omega^{S^c}(\beta_{S^c}) \leq L\Omega(\beta_S)$.*

Lemma 4.1 *Suppose S is an allowed set. Then*

$$\delta_\Omega(L, S) = \min \left\{ \frac{\|X\beta\|_n}{\Omega(\beta_S)} : \beta \text{ satisfies the } (L, S)\text{-cone condition, } \beta_S \neq 0 \right\}$$

and hence, for all β that satisfy the (L, S) -cone condition,

$$\Omega(\beta_S) \leq \Gamma_\Omega(L, S) \|X\beta\|_n.$$

The ingredients for an oracle inequality are now:

- the dual-norm equality,
- the triangle inequality,
- weak decomposability.

In other words, the situation is as for the Lasso, but the decomposability property is weakened. The dual norm of Ω is denoted by Ω_* , that is

$$\Omega_*(w) := \sup_{\Omega(\beta) \leq 1} |w^T \beta|, \quad w \in \mathbb{R}^p.$$

We moreover let $\Omega_*^{S^c}$ be the dual norm of Ω^{S^c} .

Theorem 4.1 *Let $\beta \in \mathbb{R}^p$ be arbitrary and let $S \supset \{j : \beta_j \neq 0\}$ be an allowed set. Define*

$$\lambda^S := \Omega_* \left((\epsilon^T X)_S / n \right), \quad \lambda^{S^c} := \Omega_*^{S^c} \left((\epsilon^T X)_{S^c} / n \right).$$

Suppose

$$\lambda > \lambda^{S^c}.$$

Define for some $0 \leq \delta < 1$

$$L_S := \left(\frac{\lambda + \lambda^S}{\lambda - \lambda^{S^c}} \right) \left(\frac{1 + \delta}{1 - \delta} \right).$$

Then

$$\begin{aligned} & \|X(\hat{\beta} - \beta^0)\|_n^2 + \delta(\lambda - \lambda^{S^c})\Omega^{S^c}(\hat{\beta}_{S^c}) + \delta(\lambda + \lambda^S)\Omega(\hat{\beta}_S - \beta) \\ & \leq \|X(\beta - \beta^0)\|_n^2 + \left[(1 + \delta)(\lambda + \lambda^S) \right]^2 \Gamma_\Omega^2(L_S, S). \end{aligned}$$

Theorem 4.1 requires that $S \supset S_\beta$ is an allowed set. If, for values of β that one considers as good approximations of β^0 , the smallest allowed set $S \supset S_\beta$ is much larger than S_β , then the penalty is simply not suited to describe the underlying sparsity structure.

As a special case, one may take $\beta = \beta_0$ and S_0 the smallest allowed set containing all non-zero β_j^0 ($j = 1, \dots, p$). However, the trade-off between approximation error $\|X(\beta - \beta^0)\|_n^2$ and estimation error $(\lambda + \lambda^S)^2 \Gamma_\Omega^2(L_S, S)$ will give better bounds. Theorem 4.1 is sharp as the constant in front of the approximation error $\|X(\beta - \beta^0)\|_n^2$ is one. The choice $\delta = 0$ is optimal if one only is interested in bounds for the prediction error $\|X(\hat{\beta} - \beta^0)\|_n^2$.

5 Some examples

5.1 The Lasso

The ℓ_1 -norm $\Omega(\cdot) := \|\cdot\|_1$ is (weakly) decomposable for all S , with $\Omega_{S^c} = \Omega$, and $\Omega_* = \|\cdot\|_\infty$. Hence, for all β the set S_β is an allowed set, that is, we can take $S = S_\beta$ in Theorem 4.1. The choice $\delta = 0$ then gives Theorem 3.1. For $\delta > 0$ however, we see that we also obtain a bound for $\|\hat{\beta} - \beta\|_1$, and hence for the ℓ_1 -estimation error $\|\hat{\beta} - \beta^0\|_1$. Here, one can use the triangle inequality $\|\hat{\beta} - \beta^0\|_1 \leq \|\hat{\beta} - \beta\|_1 + \|\beta - \beta^0\|_1$, i.e., it again involves a trade-off.

5.2 Group Lasso

Also the group Lasso norm $\|\cdot\|_{2,1}$ falls within the framework of decomposable norms. Let $G_t \subset \{1, \dots, T\}$, $\cup_{t=1}^T G_t = \{1, \dots, p\}$, $G_1 \cap \dots \cap G_T = \emptyset$ be a partition

of $\{1, \dots, p\}$ into disjoint groups. The norm corresponding to the group Lasso penalty is

$$\Omega(\beta) := \|\beta\|_{2,1} := \sum_{t=1}^T \sqrt{|G_t|} \|\beta_{G_t}\|_2, \quad \beta \in \mathbb{R}^p.$$

It is (weakly) decomposable for $S = \cup_{t \in \mathcal{T}} G_t$ (\mathcal{T} being any subset of $\{1, \dots, T\}$), with $\Omega_{S^c} = \Omega$. Thus, we can take $S := \cup \{G_t : \|\beta_{G_t}\|_2 \neq 0\}$ as allowed set, that is, as soon as $\beta_j \neq 0$ for some $j \in G_t$, we take the whole group of indexes G_t into our allowed set S . The dual norm is

$$\Omega_*(w) := \|w\|_{2,\infty} := \max_{1 \leq t \leq T} \|w_{G_t}\|_2 / \sqrt{|G_t|}, \quad w \in \mathbb{R}^p.$$

Let $X_{G_t} := \{X_j\}_{j \in G_t}$ be the $n \times |G_t|$ design matrix of the variables in group t ($t = 1, \dots, T$). Suppose that within groups the design is orthonormal, that is $X_{G_t}^T X_{G_t} / n = I$ for all t . Then $\|X \beta_{G_t}\|_n = \|\beta_{G_t}\|_2$ and when $\epsilon \sim \mathcal{N}(0, I)$, the random variables $\|(\epsilon^T X)_{G_t}\|_2^2$ have a χ^2 -distribution with $|G_t|$ degrees of freedom. Thus,

$$\lambda_0^2 := \|(\epsilon^T X)\|_{2,\infty}^2$$

is the maximum of T normalized χ^2 -random variables. Invoking probability inequalities for such maxima, Theorem 4.1 then gives similar (but sharp) oracle results as those in Lounici et al. [2011] or Bühlmann and van de Geer [2011].

5.3 General structured sparsity

The following example describes a general structured sparsity norm, as introduced by Micchelli et al. [2010]. Let $\mathcal{A} \subset [0, \infty)^p$ be some convex cone, satisfying $\mathcal{A} \cup (0, \infty)^p \neq \emptyset$, and

$$\Omega(\beta) := \Omega(\beta; \mathcal{A}) := \min_{a \in \mathcal{A}} \frac{1}{2} \sum_{j=1}^p \left(\frac{\beta_j^2}{a_j} + a_j \right).$$

Here we use the convention $0/0 = 0$. The assumption $\mathcal{A} \cup (0, \infty)^p \neq \emptyset$ says that there is an $a \in \mathcal{A}$ with all entries positive, so that for all β , $\Omega(\beta) < \infty$. It is shown in Micchelli et al. [2010] that Ω is indeed a norm.

Let $\mathcal{A}_S := \{a_S : a \in \mathcal{A}\}$.

Definition 5.1 *We call \mathcal{A}_S an allowed set, if*

$$\mathcal{A}_S \subset \mathcal{A}.$$

Thus we use the same terminology for sets in \mathbb{R}^p (such as \mathcal{A}_S) and index sets S .

Lemma 5.1 *Suppose \mathcal{A}_S is an allowed set. Then S is allowed, that is if we take*

$$\Omega^{S^c}(\beta_{S^c}) = \Omega(\beta_{S^c}; \mathcal{A}_{S^c}), \quad \beta_{S^c} \in \mathbb{R}^{p-|S|},$$

where $\mathcal{A}_{S^c} := \{a_{S^c} : a \in \mathcal{A}\}$, then the set S is weakly decomposable for Ω .

Note that \mathcal{A}_{S^c} is a cone and that there always is an $a_{S^c} \in \mathcal{A}_{S^c}$ which has all entries positive except for those in \mathcal{A} . Hence the restriction of $\Omega(\cdot; \mathcal{A}_{S^c})$ to $\{\beta_{S^c} : \beta \in \mathbb{R}^p\}$ is a norm. We do not require \mathcal{A}_{S^c} to be an allowed set.

Example 5.1 *As in Micchelli et al. [2010], consider the convex cone*

$$\mathcal{A} := \{a_1 \geq a_2 \geq \dots \geq a_p \geq 0\}.$$

The norm-penalty with norm $\Omega(\beta, \mathcal{A})$ then favors putting the last indexes equal to zero. Moreover, for any s , the set of the first s indexes $\{1, \dots, s\}$ is an allowed set. A partition $\{G_t\}_{t=1}^T$ is called contiguous if for all $t = 1, \dots, T-1$ and all $j \in G_t$ and $k \in G_{t+1}$ it holds that $j < k$. In Micchelli et al. [2010] it is shown that for all β there is a unique contiguous partition $\{G_t\}_{t=1}^T$ of $\{1, \dots, p\}$ such that

$$\Omega(\beta; \mathcal{A}) = \sum_{t=1}^T \sqrt{|G_t|} \|\beta_{G_t}\|_2.$$

We now return to the general norm $\Omega(\cdot; \mathcal{A})$. Its dual norm is

$$\Omega_*(w; \mathcal{A}) = \max_{a \in \mathcal{A}(1)} \sqrt{\sum_{j=1}^p a_j w_j^2}, \quad w \in \mathbb{R}^p,$$

where $\mathcal{A}(1) := \{a \in \mathcal{A} : \|a\|_1 = 1\}$. A similar expression holds for the dual norm $\Omega_*^{S^c}$ of Ω^{S^c} .

Maurer and Pontil [2012] provide moment inequalities for $\Omega_*(\epsilon^T X; \mathcal{A})$. They show that when $\epsilon \sim \mathcal{N}(0, I)$, then

$$\mathbb{E} \Omega_*(\epsilon^T X; \mathcal{A}) / n \leq \lambda_\epsilon,$$

with

$$\lambda_\epsilon := \sqrt{\frac{8}{n}} \left(2 + \sqrt{\log |\text{extreme points of } \mathcal{A}(1)|} \right) \sqrt{\frac{\sum_{i=1}^n \Omega_*^2(x_i; \mathcal{A})}{n}},$$

where $x_i = (x_{i,1}, \dots, x_{i,p})$ is the i -th row of X . Using concentration of measure (Talagrand [1995]), this can be turned into a suitable probability inequality. Again, the results can be applied to $\Omega_*^{S^c}$ as well.

The ℓ_1 -norm is a special case of the structured sparsity norm, with $\mathcal{A} = [0, \infty)^p$.

The norm $\|\cdot\|_{2,1}$ corresponding to the group Lasso, as described in Subsection 5.2 is also a special case, with

$$\mathcal{A} := \{a \in [0, \infty)^p \text{ is constant within groups}\}.$$

5.4 A trivial example

A trivial example is the norm

$$\Omega_G(\beta) := \sqrt{|G|} \|\beta_G\|_2 + \|\beta_{G^c}\|_1,$$

which is a special case of the group Lasso norm, with $n - |G| + 1$ groups, namely, the group G and $n - |G|$ groups $\{j\}_{j \notin G}$, each containing only one element. It is weakly decomposable for each $S \supset G$ with $\Omega^{S^c} = \|\cdot\|_1$. We will invoke this example mainly for facilitating our discussion of the relation between Ω -eigenvalues (see Section 6).

5.5 Overlapping groups

In this example, we consider a norm corresponding to the group Lasso with overlapping groups (Jacob et al. [2009]). Let $\{G_t\}_{t=1}^T$ be subsets of $\{1, \dots, p\}$, with $\cup_{t=1}^T G_t = \{1, \dots, p\}$, and define

$$\Omega_{\text{overlap}}(\beta) := \min \left\{ \sum_{j=1}^T \|b_t\|_2 : (b_t)_{G_t^c} = 0 \ \forall \ t, \ \sum_{t=1}^T b_t = \beta \right\}.$$

The paper Jacob et al. [2009] shows that Ω_{overlap} is indeed a norm. However, as such Ω_{overlap} is not weakly decomposable for useful candidate sets S . On the other hand by a reparametrization with parameters $\{b_t\}_{t \in T}$, we can reformulate the overlapping group Lasso problem into a group Lasso problem with non-overlapping groups. To see this, note that

$$X\beta = \sum_{t=1}^T Xb_t, \quad \sum_{t=1}^T b_t = \beta.$$

Thus, the overlapping group Lasso estimator is $\hat{\beta} = \sum_{t=1}^T \hat{b}_t$, where

$$\{\hat{b}_t\}_{t=1}^T := \arg \min_{\{b_t\}_{t=1}^T : (b_t)_{G_t^c} = 0 \ \forall \ t} \left\{ \|Y - \sum_{t=1}^T Xb_t\|_n^2 + 2\lambda \sum_{t=1}^T \|b_t\|_2 \right\}.$$

The augmented model has $\tilde{p} := \sum_{t=1}^T |G_t|$ parameters $\{b_{j,t} : j \in G_t\}_{t=1}^T$ and the augmented groups are $\tilde{G}_t := \{(j, t) : j \in G_t\}$ ($t = 1, \dots, T$), which are by definition non-overlapping. However, in the augmented design matrix

$$\tilde{X} := \left\{ \{X_j : j \in G_t\} \right\}_{t=1}^T$$

the column X_j appears $N_j := \sum_{t=1}^T \mathbf{1}\{j \in G_t\}$ times ($j = 1, \dots, p$). Although such repetitions are not a problem for the Lasso (see Remark 3.1), the implications for the group Lasso are not so clear.

6 Comparing Ω -eigenvalues

The question arises to what extent using a norm-penalty with norm Ω different from the ℓ_1 -norm results in better oracle inequalities. This partly depends on the behavior of the dual norm, a topic we briefly discuss in Section 7. It also depends on the behavior of the Ω -eigenvalues, which is the theme of the present section.

Fix a set S and consider again the norm Ω_S -defined in Section 5.4:

$$\Omega_S(\beta) = \sqrt{|S|}\|\beta_S\|_2 + \|\beta_{S^c}\|_1.$$

This norm is decomposable for S with $\Omega^{S^c} = \|\cdot\|_1$. The Ω_S -eigenvalue $\delta_{\Omega_S}(L, S)$ is the distance between the contour of the ellipse $\{X\beta_S : \|\beta_S\|_2 = 1/\sqrt{|S|}\}$ and the convex hull including interior $\{X\beta_{S^c} : \|\beta_{S^c}\|_1 \leq L\}$.

Remark 6.1 *In fact, $\delta_{\Omega_S}(L, S)$ is in part easy to compute: for fixed β_{S^c} one calculates*

$$\min_{\|\beta_S\|_2^2=1/|S|} \|X\beta_S - X\beta_{S^c}\|_n^2 := \mathcal{R}^2(\beta_{S^c}).$$

This is a quadratic minimization problem with quadratic restriction, which can be solved using Lagrange calculus. The more difficult part is to find the minimizer of $\mathcal{R}^2(\beta_{S^c})$ over all $\|\beta_{S^c}\|_1 \leq L$.

In Bühlmann and van de Geer [2011], $|S| \times \delta_{\Omega_S}^2(L, S)$ is called the *adaptive restricted eigenvalue* (because it occurred there in conjunction with the adaptive Lasso).

Recall that $\delta(L, S)$ is the ℓ_1 -eigenvalue. Since $\|\beta_S\|_1 \leq \sqrt{|S|}\|\beta_S\|_2$, one easily checks that

$$\delta(L, S) \geq \delta_{\Omega_S}(L, S),$$

i.e., the ℓ_1 -eigenvalue $\delta(L, S)$ is better behaved than the Ω_S -eigenvalue $\delta_{\Omega_S}(L, S)$.

Consider now the structured sparsity norm $\Omega(\cdot; \mathcal{A})$ introduced in Section 5.3. By Lemma 5.1, we know that under the condition that \mathcal{A}_S is allowed, the norm $\Omega(\cdot; \mathcal{A})$ is weakly decomposable for S with $\Omega^{S^c}(\beta_{S^c}) = \Omega(\beta_{S^c}; \mathcal{A}_{S^c})$.

We note that

$$\Omega(\beta; \mathcal{A}) \geq \|\beta\|_1,$$

and $\iota_S \in \mathcal{A}$, where ι is the constant vector $\iota := (1, \dots, 1)$, then

$$\Omega(\beta_S; \mathcal{A}) \leq \sqrt{|S|}\|\beta_S\|_2.$$

In other words, $\Omega(\cdot, \mathcal{A})$ intermediates the ℓ_1 - and ℓ_2 -norm.

Lemma 6.1 *Suppose \mathcal{A}_S is allowed and that $\iota_S \in \mathcal{A}$, where ι is the constant vector $\iota := (1, \dots, 1)$. Then for all $L > 0$,*

$$\delta_{\Omega}(L, S) \geq \delta_{\Omega_S}(L, S).$$

It follows that

$$\Gamma^2(L, S) \leq \Gamma_{\Omega_S}^2(L, S).$$

and more generally, under the conditions of Lemma 6.1

$$\Gamma_{\Omega}^2(L, S) \leq \Gamma_{\Omega_S}^2(L, S).$$

The Ω -effective sparsity $\Gamma_{\Omega}^2(L, S)$ is in general not comparable to the $\|\cdot\|_1$ -effective sparsity $\Gamma^2(L, S)$ for the ℓ_1 -norm $\|\cdot\|_1$. This is only partly due to the fact that the cone condition for Ω and the cone condition for $\|\cdot\|_1$ are not comparable. We finally note that the restricted eigenvalue (see Bickel et al. [2009]) is in between $|S|\delta_{\Omega_S}^2(L, S)$ and $|S|\delta^2(L, S)$, and that the Ω -eigenvalue $\delta_{\Omega}(L, S)$ is not comparable to the restricted eigenvalue either, which is now solely due to the incomparability of the cone conditions.

7 Discussion

We have shown that sparsity oracle properties hold for the least squares estimator with separable norm-penalty. There are a few issues that can be addressed here.

First of all, the choice of a norm other than $\|\cdot\|_1$ can be inspired by the practical use: the estimator may have a better interpretation. On the other hand, it may be harder to compute.

The second point is that with another norm, the dual norm may better behaved than with the ℓ_1 norm. This is the case for for instance the group Lasso, which wins in certain cases from the Lasso by a $\log p$ -term. In this paper, we have not discussed in detail the properties of the dual $\Omega_*((\epsilon^T X)_S)$ or $\Omega_*^{S^c}((\epsilon^T X)_{S^c})$ to avoid digressions. General results can be found in Maurer and Pontil [2012]. Larger norms have smaller dual norms, that is if $\Omega(\beta) \geq \tilde{\Omega}(\beta)$ for all β , then $\Omega_*(w) \leq \tilde{\Omega}_*(w)$ for all w . Note that Theorem 4.1 gives bounds for the Ω -error of $\hat{\beta}_{S_0}$, so not only its dual norm is smaller than that of $\tilde{\Omega}$, but also the bound holds for the $\tilde{\Omega}$ -error. In particular, this comparison can be made between the structured sparsity norm $\Omega(\cdot; \mathcal{A})$ defined in Section 5.3 and the ℓ_1 -norm, because $\Omega(\beta; \mathcal{A}) \geq \|\beta\|_1$ for all β . Note further that Theorem 4.1 also involves Ω^{S^c} and its dual $\Omega_*^{S^c}$, and that its result can be optimized by taking the largest possible choice for Ω^{S^c} (which will then also optimize the Ω -eigenvalue).

Of course, the prize to pay for using a norm different from ℓ_1 is that it may only be weakly decomposable for relatively large sets S . That is, one should choose a norm that corresponds to a priori knowledge on the sparsity structure.

It is to be noted further that with invoking the dual norm equality one might not exploit in full the structure of the problem. More refined techniques are given in for example van de Geer and Lederer [2012].

In cases where the penalty involves a “smoothness” norm (for example a Sobolev norm), the philosophy is again different. In the classical setup, such a penalty

is invoked for establishing (non-adaptive) smoothness only. In more recent settings, the aim is to obtain both sparsity and smoothness. An example, concerning the high-dimensional additive model, is in Meier et al. [2009]. There, the issue of decomposability, comes up as well. Oracle results are derived using a penalty that is not only sparsity decomposable but also “smoothness” decomposable (see also Bühlmann and van de Geer [2011], Section 8.4.5).

Finally, the oracle results can be extended to loss functions other than least squares (for example in the spirit of van de Geer [2008] or Negahban et al. [2012]). Sharp oracle results are discussed in van de Geer [2013]. For the quasi-likelihood loss with canonical link function, the dual-norm argument can again be used. For other cases this argument generally has to be replaced. Here, tools from empirical process theory can be invoked (such as those outlined in Bühlmann and van de Geer [2011], Chapter 8).

8 Proofs

Proof of Lemma 4.1. Let $\beta \in \mathcal{C} := \{\Omega^{Sc}(\beta_{Sc}) \leq L\Omega(\beta_S) \neq 0\}$. Write

$$\tilde{\beta}_S := \frac{\beta_S}{\Omega(\beta_S)}, \quad \tilde{\beta}_{Sc} := \frac{\beta_{Sc}}{\Omega(\beta_S)}.$$

Then $\Omega(\tilde{\beta}_S) = 1$ and $\Omega^{Sc}(\tilde{\beta}_{Sc}) \leq L$, and hence

$$\frac{\|X\beta\|_n}{\Omega(\beta_S)} = \|X\tilde{\beta}_S + X\tilde{\beta}_{Sc}\|_n.$$

It follows that

$$\min_{\beta \in \mathcal{C}} \frac{\|X\beta\|_n}{\Omega(\beta_S)} = \delta_\Omega(L, S).$$

□

The next lemma shows why convexity of the penalty is important. The result can be extended to loss functions other than quadratic loss, see the rejoinder in the discussion paper van de Geer [2013].

Lemma 8.1 *Let \mathcal{B} be a convex subset of \mathbb{R}^p and $\text{pen} : \mathcal{B} \rightarrow \mathbb{R}$ be a convex penalty. Let moreover*

$$\hat{\beta} := \arg \min_{\beta \in \mathcal{B}} \|Y - X\beta\|_n^2 + 2\text{pen}(\beta).$$

Then for every $\beta \in \mathcal{B}$

$$(Y - X\hat{\beta})^T X(\beta - \hat{\beta})/n + \text{pen}(\hat{\beta}) \leq \text{pen}(\beta).$$

Proof. Fix $\beta \in \mathcal{B}$ and define for $0 < \alpha \leq 1$,

$$\hat{\beta}_\alpha := (1 - \alpha)\hat{\beta} + \alpha\beta.$$

We have

$$\begin{aligned} \|Y - X\hat{\beta}\|_n^2 + 2\text{pen}(\hat{\beta}) &\leq \|Y - X\hat{\beta}_\alpha\|_n^2 + 2\text{pen}(\hat{\beta}_\alpha) \\ &\leq \|Y - X\hat{\beta}_\alpha\|_n^2 + 2(1 - \alpha)\text{pen}(\hat{\beta}) + 2\alpha\text{pen}(\beta) \end{aligned}$$

where we used the convexity of the penalty. It follows that

$$\frac{\|Y - X\hat{\beta}\|_n^2 - \|Y - X\hat{\beta}_\alpha\|_n^2}{\alpha} + 2\text{pen}(\hat{\beta}) \leq 2\text{pen}(\beta).$$

But clearly

$$\lim_{\alpha \downarrow 0} \frac{\|Y - X\hat{\beta}\|_n^2 - \|Y - X\hat{\beta}_\alpha\|_n^2}{\alpha} = 2(Y - X\hat{\beta})^T X(\beta - \hat{\beta})/n.$$

□

Proof 4.1. Let us write for $v, w \in \mathbb{R}^n$,

$$(v, w) := v^T w / n.$$

Fix some $\beta \in \mathbb{R}^p$ and let $S \supset \{j : \beta_j \neq 0\}$ be an allowed set. If

$$(X(\hat{\beta} - \beta^0), X(\hat{\beta} - \beta))_n \leq -(\delta(\lambda + \lambda^S)\Omega(\hat{\beta}_S - \beta) + \delta(\lambda - \lambda^{S^c})\Omega(\hat{\beta}_{S^c}))$$

we find

$$\begin{aligned} &\|X(\hat{\beta} - \beta^0)\|_n^2 + \delta(\lambda + \lambda^S)\Omega(\hat{\beta}_S - \beta) + \delta(\lambda - \lambda^{S^c})\Omega(\hat{\beta}_{S^c}) - \|X(\beta - \beta^0)\|_n^2 \\ &= \delta(\lambda + \lambda^S)\Omega(\hat{\beta}_S - \beta) + \delta(\lambda - \lambda^{S^c})\Omega(\hat{\beta}_{S^c}) - \|X(\beta - \hat{\beta})\|_n^2 + 2(X(\hat{\beta} - \beta^0), X(\hat{\beta} - \beta))_n \leq 0. \end{aligned}$$

Hence, then we are done.

Suppose now that

$$(X(\hat{\beta} - \beta^0), X(\hat{\beta} - \beta))_n \geq -(\delta(\lambda + \lambda^S)\Omega(\hat{\beta}_S - \beta) + \delta(\lambda - \lambda^{S^c})\Omega(\hat{\beta}_{S^c})).$$

By Lemma 8.1 we have

$$((Y - X\hat{\beta}), X(\beta - \hat{\beta}))_n + \lambda\Omega(\hat{\beta}) \leq \lambda\Omega(\beta),$$

or

$$(X(\hat{\beta} - \beta^0), X(\hat{\beta} - \beta))_n + \lambda\Omega(\hat{\beta}) \leq (\epsilon, X(\hat{\beta} - \beta))_n + \lambda\Omega(\beta).$$

By definition of the dual norm,

$$(\epsilon, X(\hat{\beta} - \beta))_n = (\epsilon, X(\hat{\beta}_S - \beta))_n + (\epsilon, X(\hat{\beta}_{S^c} - \beta))_n \leq \lambda^S\Omega(\hat{\beta}_S - \beta) + \lambda^{S^c}\Omega^{S^c}(\hat{\beta}_{S^c}).$$

Thus

$$(X(\hat{\beta} - \beta^0), X(\hat{\beta} - \beta))_n + \lambda\Omega(\hat{\beta}) \leq \lambda^S\Omega(\hat{\beta}_S - \beta) + \lambda^{S^c}\Omega^{S^c}(\hat{\beta}_{S^c}) + \lambda\Omega(\beta).$$

By the weak decomposability of Ω and the triangle inequality, this implies

$$(X(\hat{\beta} - \beta^0), X(\hat{\beta} - \beta))_n + (\lambda - \lambda^{S^c})\Omega^{S^c}(\hat{\beta}_{S^c}) \leq (\lambda + \lambda^S)\Omega(\hat{\beta}_S - \beta). \quad (1)$$

Since $(X(\hat{\beta} - \beta^0), X(\hat{\beta} - \beta))_n \geq -(\delta(\lambda + \lambda^S)\Omega(\hat{\beta}_S - \beta) + \delta(\lambda - \lambda^{S^c})\Omega^{S^c}(\hat{\beta}_{S^c}))$ this gives

$$\Omega^{S^c}(\hat{\beta}_{S^c}) \leq L_S \Omega(\hat{\beta}_S - \beta).$$

We now insert Lemma 4.1, which gives

$$\Omega(\hat{\beta}_S - \beta) \leq \Gamma_\Omega(L, S) \|X(\hat{\beta} - \beta)\|_n \quad (2)$$

and continue with inequality (1):

$$\begin{aligned} & (X(\hat{\beta} - \beta^0), X(\hat{\beta} - \beta))_n + (\lambda - \lambda^{S^c})\Omega(\hat{\beta}_{S^c}) + \delta(\lambda + \lambda^S)\Omega(\hat{\beta}_S - \beta) \\ & \leq [(1 + \delta)(\lambda + \lambda^S)]\Gamma_\Omega(L_S, S) \|X(\hat{\beta} - \beta)\|_n \\ & \leq \frac{1}{2} \left[(1 + \delta)(\lambda + \lambda^S) \right]^2 \Gamma_\Omega^2(L_S, S) + \frac{1}{2} \|X(\hat{\beta} - \beta)\|_n^2. \end{aligned}$$

Since

$$2(X(\hat{\beta} - \beta^0), X(\hat{\beta} - \beta))_n = \|X(\hat{\beta} - \beta^0)\|_n^2 - \|X(\beta - \beta^0)\|_n^2 + \|X(\beta - \hat{\beta})\|_n^2,$$

we obtain

$$\begin{aligned} & \|X(\hat{\beta} - \beta^0)\|_n^2 + 2(\lambda - \lambda^{S^c})\Omega(\hat{\beta}_{S^c}) + 2\delta(\lambda + \lambda^S)\Omega(\hat{\beta}_S - \beta) \\ & \leq \|X(\beta - \beta^0)\|_n^2 + \left[(1 + \delta)(\lambda + \lambda^S) \right]^2 \Gamma_\Omega^2(L_S, S). \end{aligned}$$

□

Proof of Lemma 5.1. Note that for any a and β

$$\frac{1}{2} \sum_{j=1}^p \left(\frac{\beta_j^2}{a_j} + a_j \right) = \frac{1}{2} \sum_{j \in S} \left(\frac{\beta_j^2}{a_j} + a_j \right) + \frac{1}{2} \sum_{j \in S^c} \left(\frac{\beta_j^2}{a_j} + a_j \right).$$

Hence, writing

$$a(\beta) := \arg \min_{a \in \mathcal{A}} \frac{1}{2} \sum_{j=1}^p \left(\frac{\beta_j^2}{a_j} + a_j \right),$$

we have

$$\begin{aligned} \Omega(\beta) &:= \frac{1}{2} \sum_{j=1}^p \left(\frac{\beta_j^2}{a_j(\beta)} + a_j(\beta) \right) \\ &= \frac{1}{2} \sum_{j \in S} \left(\frac{\beta_j^2}{a_j(\beta)} + a_j(\beta) \right) + \frac{1}{2} \sum_{j \in S^c} \left(\frac{\beta_j^2}{a_j(\beta)} + a_j(\beta) \right) \\ &= \frac{1}{2} \sum_{j=1}^p \left(\frac{\beta_{j,S}^2}{a_{j,S}(\beta)} + a_{j,S}(\beta) \right) + \frac{1}{2} \sum_{j=1}^p \left(\frac{\beta_{j,S^c}^2}{a_{j,S^c}(\beta)} + a_{j,S^c}(\beta) \right) \\ &\geq \min_{a_S \in \mathcal{A}_S} \frac{1}{2} \sum_{j=1}^p \left(\frac{\beta_{j,S}^2}{a_{j,S}} + a_{j,S} \right) + \min_{a_{S^c} \in \mathcal{A}_{S^c}} \frac{1}{2} \sum_{j=1}^p \left(\frac{\beta_{j,S^c}^2}{a_{j,S^c}} + a_{j,S^c} \right) \end{aligned}$$

$$\geq \Omega(\beta_S) + \Omega^{S^c}(\beta_{S^c}).$$

□

Proof of Lemma 6.1. Suppose β satisfies the (L, S) -cone condition for Ω :

$$\Omega^{S^c}(\beta_{S^c}) \leq L\Omega(\beta_S),$$

then also

$$\Omega_S^{S^c}(\beta_{S^c}) = \|\beta_{S^c}\|_1 \leq \Omega^{S^c}(\beta_{S^c}) \leq L\Omega(\beta_S) \leq L\sqrt{|S|}\|\beta_S\|_2 = L\Omega_S(\beta_S),$$

where in the last inequality we used $\iota_S \in \mathcal{A}$. Hence, β satisfies the (L, S) -cone condition for Ω_S . But then

$$\Omega(\beta_S) \leq \Omega_S(\beta_S) \leq \frac{\|X\beta\|_n}{\delta_{\Omega_S}(L, S)}.$$

The result now follows from Lemma 4.1.

□

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